Solutions to Problem Set 1 due September 9, at 10 AM, on Gradescope (via Stellar)

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue

Problem 1: Find a 2×2 matrix A and a 3×3 matrix B such that:

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 5x + y \end{bmatrix} \qquad \text{and} \qquad B \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ z \\ y \end{bmatrix}$$

for any numbers x, y, z. *(25 points)*

Solution: We rewrite the right-hand sides in matrix form as follows:

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and

$$B \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Grading Rubric:

- Correct matrix A *(10 points)*
- Slightly wrong matrix A ie transposed or some entries swapped *(5 points)*
- Missing or totally incorrect matrix A *(0 points)*
- Correct matrix B *(15 points)*
- Slightly wrong matrix B ie transposed or some entries swapped *(10 points)*
- Missing or totally incorrect matrix B *(0 points)*

Problem 2: What is the largest number n of vectors in the xy–plane, such that the dot product of any two of them is **strictly** negative? Argue by showing a collection of n vectors which have this property, and argue why there cannot exist a collection of $n + 1$ vectors with this property. *(25 points)*

Solution: Two vectors have negative dot product if the angle between them is obtuse (i.e. $> 90°$). There cannot be 4 vectors in the plane, the angle between any two of them being $> 90°$, because the total angle around the origin in the plane is $360°$. So the maximal number of such vectors is $n = 3$, and indeed we can find 3 vectors in the plane with all obtuse angles between them (example $(1, 0)$, $(-1, 2)$, $(-1, -2)$ or many other choices). We also accept a picture of 3 vectors, with all obtuse angles between them, if the person makes reference to the obtuseness of the angles.

Note: we will accept any other idea that leads to a correct proof, including one which just writes out a bunch of vectors in components, and works out conditions on those components that correspond to the dot products being negative.

Grading Rubric:

- Correct 3 vectors satisfying conditions *(15 points)*
- Slightly wrong 3 vectors satisfying conditions (e.g. some of the angles not $> 90°$) *(10 points)*
- Totally wrong 3 vectors satisfying conditions or different number of vectors *(0 points)*
- Correct argument excluding the possibility of 4 vectors *(10 points)*
- Partial progress in argument excluding 4 vectors *(5 points)*
- Completely incorrect argument for excluding the case of 4 vectors *(0 points)*

Please assign at least 10 points to any student who identifies that negative dot product is equivalent to angle being obtuse, and at least 5 points to any student who recalls that dot product measures the angle between vectors.

Problem 3: Consider the matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ e & 1 & 0 \\ \sqrt{2} & \pi & 1 \\ 10^{-1} & 10^{-2} & 10^{-3} \end{bmatrix}$$

and put it in row echelon form. *(25 points)*

Solution: Note that this matrix has 1's on the diagonal and 0 above it, so using Gaussian elimination will easily give:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{1}$$

regarless of what wacky entries you put under the diagonal of 1's. For example, subtracting e times row 1 from row 2 will give you:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \sqrt{2} & \pi & 1 \\ 10^{-1} & 10^{-2} & 10^{-3} \end{bmatrix}$$

and then subtracting $\sqrt{2}$ times row 1 from row 3 will give you:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \pi & 1 \\ 10^{-1} & 10^{-2} & 10^{-3} \end{bmatrix}$$

Repeating this argument a number of times (specifically 6 times) produces the row echelon form (1).

Grading Rubric:

- Correct argument with or without direct computation *(25 points)*
- Correct answer without argument *(15 points)*
- Computation with mistakes, but correct answer *(20 points)*
- Computation with mistakes, but significantly wrong answer *(10 points)*
- Wrong Gaussian elimination process or no answer *(0 points)*

Problem 4: Consider Gaussian elimination for the following system of equations:

$$\begin{cases} x - y + 2z = 1 \\ -2x + \lambda y - z = 0 \\ -3x + 2y - 5z = 0 \end{cases}$$

for some number λ.

- For which choice of λ does Gaussian elimination require you to swap the second and third rows? Solve the system for that value of λ. *(15 points)*
- For which choice of λ is the matrix singular? *(10 points)*

Solution: We start by translating the problem in matrix notation to get:

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & \lambda & -1 \\ -3 & 2 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Now we do Gaussian elimination using the first row to eliminate entries in the first column to get:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & \lambda - 2 & 3 & 2 \\ 0 & -1 & 1 & 3 \end{array}\right]$$

Note that we can only use the second row for elimination if the pivot in the second row is not 0, ie Gaussian elimination requires a swap if $\lambda = 2$. This is the answer for the first bullet. In this case, after swapping rows we get the equivalent system given by:

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

So by back substitution we get the solution is $z = \frac{2}{3}$, $y = -\frac{7}{3}$ and $x = -\frac{8}{3}$
For the second part, note that for $\lambda = 2$ the matrix is not singular. Meanwhile, if $\lambda \neq 2$, continuing Gaussian elimination gives us:

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & \lambda - 2 & 3 \\ 0 & 0 & \frac{\lambda+1}{\lambda-2} \end{bmatrix}$$

So we get the matrix is singular when the last entry is 0, ie when $\lambda = -1$

Grading Rubric:

- Correct Gaussian elimination and correct solutions in first part *(15 points)*
- Slightly wrong Gaussian elimination or solutions *(10 points)*
- Correct Gaussian elimination, but missunderstanding in the row swap *(5 points)*
- Incorrect Gaussian elimination process *(0 points)*
- Correct result for the second part *(10 points)*
- Slightly wrong computation for the second part *(5 points)*
- Missing or misunderstanding the second part *(0 points)*